DIVISION OF THE HUMANITIES AND SOCIAL SCIENCES
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA 91125

WHEN DOES AGGREGATION REDUCE UNCERTAINTY AVERSION?

Christopher P. Chambers and Federico Echenique

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Abstract

We study the problem of uncertainty sharing within a household: "risk sharing," in a context of Knightian uncertainty. A household shares uncertain prospects using a social welfare function. We characterize the social welfare functions such that the household is collectively less averse to uncertainty than each member, and satisfies the Pareto principle and an independence axiom. We single out the sum of certainty equivalents as the unique member of this family which provides quasiconcave rankings over risk-free allocations.

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1 Introduction

We study when a household of uncertainty-averse individuals will behave in the aggregate in a fashion that is *less averse to uncertainty* than each member of the household.[1] The notion that a group is less uncertainty averse than its members is a very familiar one in economics. Early arguments for the notion are in Samuelson (1964), Vickrey (1964) and Arrow and Lind (1970). We characterize the households that collectively are less uncertainty averse than its members.

We consider a household of individuals who collectively face some aggregate subjective uncertainty in the consumption of a single good, say money. Members of the household have differing attitudes toward uncertainty. The household seeks to allocate the uncertainty to its members to maximize some notion of social welfare.

Our main results can be roughly stated as follows: Suppose that the household ranks allocations using a social welfare function $\succeq^0 (\succeq_1, \dots \succeq_n)$, which depends on individual preferences $\succeq_i$. The *sum of individual certainty equivalents* represents the only social welfare function (SWF) that

1. generates less uncertainty averse households (for all individual preference profiles);
2. ignores preferences over uncertainty whenever it compares uncertainty-free prospects;

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[1] We use "household" throughout to refer generically to a group of agents engaged in an uncertainty sharing arrangement.

3. is quasiconcave over uncertainty-free prospects.

This result is a simple consequence of a theorem stating that (1) and (2) are equivalent to ranking allocations with a function which takes as input certainty equivalents, and when maximized over a simplex, has a solution at every corner. This theorem, in turn, uses some classical results on aggregation from demand theory (concretely Samuelson (1956) and Chipman and Moore (1979)).

The sum of certainty equivalents has a simple economic interpretation: it is the certain monetary sum that the household would demand to accept an allocation of uncertain prospects. In other words, it is the collective willingness to pay for an allocation. Myerson (2004), for example, recommends MBAs and applied decision makers use the sum of certainty equivalents to share risk. Our paper is a justification for this type of normative recommendation.

In the specialized setting of homothetic preference profiles (such as CRRA preferences), we find another justification for the sum of certainty equivalents criterion. It yields the most uncertainty averse *convex* household preference, among those which are less uncertainty averse than members' preferences.

We proceed to discuss our results in more detail.

We suppose a finite set of states of the world. Agents' preferences are over state-contingent monetary payoffs, which we call acts. We impose little structure on the preferences of agents; in particular, agents may not be subjective expected utility maximizers. We assume that preferences are convex–if two acts are deemed indifferent, then any convex combination of the two is at least as good as each of them. Intuitively, the convex combination of the two acts represents a *hedge* against uncertainty and thus improves the individual's welfare.

A SWF recommends a preference over allocations for any given list of individual preferences. We impose the Pareto principle: the SWF must be monotone increasing in the welfare of individuals.

We present two new axioms. The first axiom is an independence axiom: the ranking of any two allocations which involve no uncertainty should be independent of the agents' uncertainty preferences. An allocation involves no uncertainty when each agent is allocated a constant act–an act whose payoff does not depend on the state. Note that acts involve monetary payoffs; hence comparing two allocations free of uncertainty simply involves a tradeoff in which agents receive more or less money. The axiom states that the tradeoff should be resolved without regard for the agents' preferences over uncertainty.

Our final and main axiom is reduction of uncertainty aversion. It says that the household is less averse to uncertainty than its members. The axiom requires defining a comparative notion of uncertainty aversion. Following Yaari (1969), we say that a preference $\succeq_1$ is less uncertainty averse than $\succeq_2$ if, for every constant act c and every act x, when $x \succeq_2 c$ then $x \succeq_1 c$. The idea is that if the uncertainty involved in choosing the uncertain act x over the certain act c is acceptable for $\succeq_2$, then it must also be acceptable for $\succeq_1$. For two subjective expected utility agents with the same prior beliefs, one agent is more uncertainty averse than the other if and only if he is more risk averse than the other.

The social welfare function guides the household's decisions on how to share uncertain prospects. In a classic paper discussing the representative consumer problem in demand theory, Samuelson (1956) shows that a household which allocates aggregate bundles optimally according to some SWF behaves as if it is an individual (that is, it has a complete and transitive preference). Our social welfare function generates such a preference for each list of individual preferences–this is what we call the *household preference*. Our main axiom requires that this household preference be less uncertainty averse than the preferences of each member of the household.

We characterize completely the family of all SWF's satisfying these properties. The simplest member of the family to explain is the unique one which is *quasiconcave over uncertainty-free prospects*. Under this assumption, the unique SWF satisfying the axioms is representable by the function, which for any allocation, returns the sum of certainty equivalents. That is, for each individual's state-contingent consumption, the rule finds the certain amount that the individual would need to be given in compensation, then adds these across individuals.

Section 2 provides the model; Section 3 has the main results; Section 4 presents results for homothetic preferences, and examples of familiar special cases. Section 5 provides discussion and related literature.

2 The model

Let Ω be a finite set of states of the world. Acts are payoff-contingent elements of $\mathbb{R}_+$; that is, the set of acts is $X = \mathbb{R}_+^\Omega$. Let N be a finite set of agents. An **allocation** is an element of X^N. An allocation of $x \in X$ is a vector $(x_i)_{i \in N} \in X^N$ for which $\sum_{i \in N} x_i = x$.

A preference relation $\succeq$ is a complete, transitive, convex, continuous, and monotonic[2] binary relation on X. The set of preferences is denoted $\mathcal{R}$. Convexity captures the property of uncertainty aversion, as first formulated by Yaari (1969).

Our aim in this study is to understand methods of aggregating preferences which reduce uncertainty aversion. We imagine a set of agents who reside in a household and use some social welfare function to optimally distribute resources. Samuelson (1956) observed that such optimization leads to "rational" behavior in the aggregate. We ask when such household behavior is less uncertainty averse than the behavior of each individual in the household.

To this end, we discuss a comparative notion of uncertainty aversion and a domain of preferences on which this exercise becomes meaningful. For $c \in \mathbb{R}_+$, we abuse notation and identify c with the constant act whose outcome in every state is c. Let $\succeq'$ and $\succeq$ be two preference relations. As in Yaari (1969), we say that $\succeq$ **is more uncertainty averse than** $\succeq'$ if for all $c \geq 0$, $\{x : x \succeq c\} \subseteq \{x : x \succeq' c\}$. Every uncertain prospect which is preferable to c by $\succeq$ is also preferable to c by $\succeq'$.

The idea that a household should be less uncertainty averse than its members has both normative and strategic content. For example, often in strategic interaction, all else equal, an agent who is less uncertainty averse will fare better according to all preferences. This is the case in Nash bargaining Rubinstein, Safra, and Thomson (1992), as well as in many other game theoretic models of bargaining. By sharing uncertainty appropriately, a household seeks to become more "competitive." Secondly, it is somehow understood that risk neutral preferences (linear preferences) are an "ideal" preference that should be strived for–models of asset pricing and finance for example deal only with the expected values of assets and not their expected utilities.

For any $\succeq \in \mathcal{R}$, define the set of **priors at constant act** c by $P(\succeq, c) =$

[2]That is, if $x(\omega) > y(\omega)$ for all $\omega \in \Omega$, then $x \succ y$.

$\{p \in \Delta(\Omega) : x \succeq c \Longrightarrow p \cdot x \geq p \cdot c\}$. The set of **priors** is

$$P(\succeq) = \bigcap_{c \geq 0} P(\succeq, c).$$

Thus, a prior at c is a probability measure over Ω such that the expectation of x dominates c; in demand theory terms, it simply represents the marginal rate of substitution given certain consumption c. A profile $(\succeq_i)_{i \in N}$ is a **common prior profile** at c if

$$\bigcap_{i \in N} P(\succeq_i, c) \neq \varnothing.$$

Common priors have no normative content in our work. The main point here is to uncover conditions which force a household to have less uncertainty aversion than each of its members. A necessary and sufficient condition for a preference to exist which is less uncertainty averse than each individual preference $((\succeq_i))_{i \in N}$ is that for all c there be some common prior at c.

We consider the stronger requirement that there be a common prior for all individuals which is a common prior at any constant act. That is, we work with the profiles $((\succeq_i))_{i \in N}$ with

$$\bigcap_{i \in N} P(\succeq_i) \neq \varnothing.$$

We denote the resulting set of profiles by $\mathcal{CP}$. Subjective expected utility profiles with a common prior are in $\mathcal{CP}$, as are many other examples.

A *domain* $\mathcal{D}$ is a subset of $\mathcal{R}^N$. A social welfare function is a mapping which carries $\mathcal{D}$ into binary relations over X^N. Formally, we denote the set of binary relations over X^N by $\mathcal{R}_N$. Then a **social welfare function** is a function $\succeq^0$: $\mathcal{D} \to \mathcal{R}_N$. We write $\succeq^0 \left((\succeq_i)_{i \in N}\right)$ for the binary relation over allocations obtained when individual preferences are $(\succeq_i)_{i \in N}$.

Example 1: A classical domain of preferences is the domain of risk averse expected utility profiles with a common prior. We denote this domain by $\mathcal{EU}$. Formally, $(\succeq_i)_{i \in N} \in \mathcal{EU}$ if there exists $p \in \Delta(\Omega)$ and for all $i \in N$, there exists $u_i : \mathbb{R}_+ \to \mathbb{R}$ which is strictly increasing, concave, and continuous for which for all $i \in N$ and all $x, y \in X$, $x \succeq_i y$ if and only if

$$\sum_{\omega \in \Omega} p(\omega) u_i(x(\omega)) \geq \sum_{\omega \in \Omega} p(\omega) u_i(y(\omega)).$$

A standard example of a social welfare function $\succeq^0: \mathcal{EU} \to \mathcal{R}_N$ used in the theory of risk sharing (for example, Borch (1962) and Wilson (1968)) is the **utilitarian rule**. For any $(\succeq_i)_{i\in N} \in \mathcal{EU}$, there exists for all $i \in N$ a unique $u_{\succeq_i} : \mathbb{R}_+ \to \mathbb{R}$ which represents $\succeq_i$ and which is normalized so that $u_{\succeq_i}(0) = 0$ and $u_{\succeq_i}(1) = 1$. We then require $(x_1, ..., x_n) \succeq^0 \left((\succeq_i)_{i\in N}\right)(y_1, ..., y_n)$ if and only if

$$\sum_{i\in N}\sum_{\omega\in\Omega} p(\omega)\, u_{\succeq_i}(x_i(\omega)) \geq \sum_{i\in N}\sum_{\omega\in\Omega} p(\omega)\, u_{\succeq_i}(y_i(\omega)).$$

Note that, while we have restricted elements of $\mathcal{R}$ significantly, elements of $\mathcal{R}_N$ obey no restrictions whatsoever (we will later make assumptions on these elements in the form of axioms). This is because elements of $\mathcal{R}$ are understood to be descriptive, whereas elements of $\mathcal{R}_N$ are normative recommendations for a society.

3 Results

We proceed to describe the four axioms. The axioms will be equivalent to a certain class of SWF. Coupled with quasiconcavity over uncertainty-free prospects, we characterize the SWF which is represented as the sum of certainty equivalents.

Our first axiom states that household preferences over allocations should be minimally "rational."

Rationality: *For all $(\succeq_i)_{i\in N} \in \mathcal{D}$, $\succeq^0 \left((\succeq_i)_{i\in N}\right)$ is continuous, monotonic, complete, and transitive.*

Our second axiom is natural and requires that the SWF comply with the Pareto property.

Pareto: *For all $(\succeq_i)_{i\in N} \in \mathcal{R}^N$ and all $(x_i)_{i\in N}, (y_i)_{i\in N} \in X^N$, if $x_i \succeq_i y_i$ for all $i \in N$, then $x \succeq^0 \left((\succeq_i)_{i\in N}\right) y$ (with strict preference if all individual preferences are strict).*

Our next axiom is the first that deals specifically with the interpretation of uncertainty. It requires that in ranking profiles of certain prospects, the social welfare function should ignore attitudes toward uncertainty. We often refer to the axiom simply as *independence.*

Independence of uncertainty attitudes for constant acts: *For all* $(\succeq_i)_{i\in N}, (\succeq_i')_{i\in N} \in \mathcal{D}$ *and all constant* $(c_i)_{i\in N}, (d_i)_{i\in N} \in X^N$, $(c_i)_{i\in N} \succeq^0 \left((\succeq_i)_{i\in N}\right)(d_i)_{i\in N} \Longleftrightarrow (c_i)_{i\in N} \succeq^0 \left((\succeq_i')_{i\in N}\right)(d_i)_{i\in N}$.

Acts are monetary lotteries, and individual preferences are monotonic, so all individual preferences coincide over constant act: more is better. The independence axiom says that, when comparing constant acts, $\succeq^0$ should not depend on individual preferences as these do not differ in the comparison of constant acts.

To understand our last axiom, we need to discuss the notion of uncertainty sharing. It is the standard notion of risk sharing, in an environment of possibly non-expected-utility maximizers (see e.g. Epstein (2001) and Rigotti and Shannon (2005)).

The individuals in N are all members of a household. Household members entertain different attitudes toward uncertainty. We imagine that the household uses a SWF to allocate an aggregate bundle x among its members. That is, given household preferences $(\succeq_i)_{i\in N}$ by maximizing $\succeq^0 \left((\succeq_i)_{i\in N}\right)$ across $\left\{(x_i)_{i\in N} \in X^N : \sum_{i\in N} x_i \leq x\right\}$. Under our continuity assumptions, this uncertainty-sharing maximization problem is well-defined. Uncertainty sharing generates a well-defined household preference over acts: an "aggregate preference." This aggregation results from a well-known aggregation result in classical demand theory (see Samuelson (1956) and Chipman and Moore (1979)).

Household preferences over acts are given by $x \succeq^* \left((\succeq_i)_{i\in N}\right) y$ if and only if for all $(y_i)_{i\in N} \in X^N$ such that $\sum_{i\in N} y_i \leq y$, there exists $(x_i)_{i\in N}$ such that $\sum_{i\in N} x_i \leq x$ and $(x_i)_{i\in N} \succeq^0 \left((\succeq_i)_{i\in N}\right)(y_i)_{i\in N}$. This binary relation is the **household preference**.

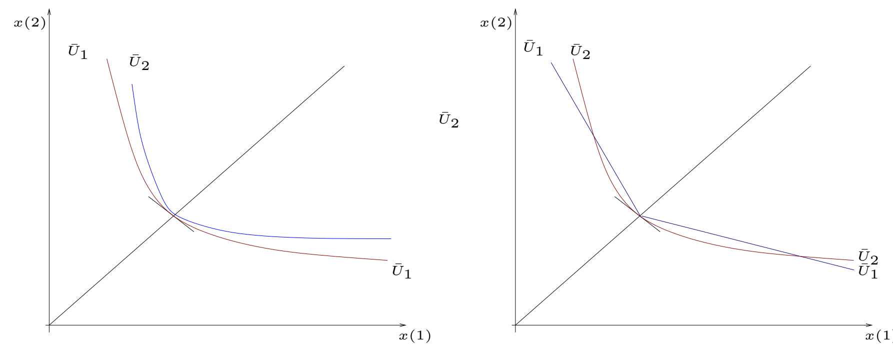
We are now ready to state our next axiom.

Reduction of uncertainty aversion: *For all* $(\succeq_i)_{i\in N} \in \mathcal{D}$, $\succeq^* \left((\succeq_i)_{i\in N}\right)$ *is less uncertainty averse than* $\succeq_i$ *for all* $i \in N$.

Example 2: Suppose $\mathcal{D} = \mathcal{EU}$ and consider the utilitarian rule defined above. In general, for any $(\succeq_i)_{i\in N} \in \mathcal{EU}$, the household preference $\succeq^* \left((\succeq_i)_{i\in N}\right)$ is expected utility with von Neumann Morgenstern utility index given by

$$u(x) = \sup_{\sum_{i\in N} x_i = x} \sum_{i\in N} u_{\succeq_i}(x_i).$$

The function u is referred to as the "sup-convolution" of the functions $u_{\succeq_i}$. It is easily verified that the resulting $\succeq^*$ may be less or more risk averse than the individual preferences.



(a) Preference $\succeq_1$ is less uncertainty averse than $\succeq_2$.

(b) Household preference must contain lower envelope of $\bar{U}_1 - \bar{U}_1$ and $\bar{U}_2 - \bar{U}_2$.

Figure 1: Comparisons in uncertainty aversion.

In general, there may not exist a binary relation which is less uncertainty averse than a profile of binary relations. But they exist for profiles in $\mathcal{D} \subseteq \mathcal{CP}$.

Figure 1 presents the problem geometrically. One preference $\succeq_1$ is less uncertainty averse than $\succeq_2$ if, when we compare their upper contour sets through any uncertainty-free act, the upper contour set of $\succeq_1$ contains that of $\succeq_2$. Figure 1(a) illustrates how the preferences having the $\bar{U}_1 - \bar{U}_1$ indifference curve are less averse to uncertainty than those having the $\bar{U}_2 - \bar{U}_2$ curve. Note that there is a common prior, indicated by the tangent line. In Figure 1(a), any household preference satisfying the reduction of uncertainty aversion axiom will need to have indifference curves "below" $\bar{U}_1 - \bar{U}_1$.

Figure 1(b) presents a case where the preferences do not have comparable uncertainty aversion. In the figure, the most uncertainty averse household preference which is less uncertainty averse than each individual agent is given by the lower envelope of the two indifference curves (i.e. by the curve which goes from $\bar{U}_1$ to the intersection of the two indifference curves, then coincides with $\bar{U}_2 - \bar{U}_2$ until the second intersection, then coincides with $\bar{U}_1 - \bar{U}_1$). Note that this household preference is not convex, and any household preference satisfying reduction in uncertainty aversion must have indifference curves below this lower envelope of $\bar{U}_1 - \bar{U}_1$ and $\bar{U}_2 - \bar{U}_2$.

For a preference $\succeq \in \mathcal{R}$, the **certainty equivalent** $ce_{\succeq} : X \to \mathbb{R}$ carries each act x to the constant act c for which $c \sim x$. By our continuity and monotonicity assumptions, certainty equivalents exist and are unique. Critically for us, for a given $\succeq$, $ce_{\succeq}$ is a continuous utility representation of $\succeq$. The following is immediate. We say a function

$W:\mathbb{R}_+^N \to \mathbb{R}$ is strictly monotonic if $x, y \in \mathbb{R}_+^N$ and $x \geq y$ implies $W(x) \geq W(y)$, and $x \gg y$ ($x_i > y_i$ for all $i \in N$) implies $W(x) > W(y)$.

Remark 1 *For two preferences $\succeq, \succeq' \in \mathcal{R}$, $\succeq$ is more uncertainty averse than $\succeq'$ if and only if for all $x \in X$, $ce_{\succeq}(x) \leq ce_{\succeq'}(x)$.*

Proposition 3: *A social welfare function on $\mathcal{D}$ satisfies rationality, Pareto, and independence if and only if there exists a strictly monotonic, continuous function $W:\mathbb{R}_+^N \to \mathbb{R}$ for which for all $(\succeq_i)_{i\in N} \in \mathcal{R}^N$ and all $(x_i)_N, (y_i)_N \in X^N$,*

$$(x_i) \succeq^0 \left((\succeq_i)_{i\in N}\right)(y_i) \Longleftrightarrow W\left(\left(ce_{\succeq_i}(x_i)\right)\right) \geq W\left(\left(ce_{\succeq_i}(y_i)\right)\right).$$

Proof: Let $(\succeq_i')_{i\in N} \in \mathcal{D}$. Define $W:\mathbb{R}^N \to \mathbb{R}$ so that $W\left((d_i)_{i\in N}\right) \geq W\left((c_i)_{i\in N}\right)$ if and only if

$$(d_i)_{i\in N} \succeq^0 \left((\succeq_i')_{i\in N}\right)(c_i)_{i\in N}.$$

W is well-defined as $\succeq^0 \left((\succeq_i')_{i\in N}\right)$ is continuous (Debreu, 1964). W is strictly monotonic by the Pareto property. Now, let $(\succeq_i)_{i\in N} \in \mathcal{R}^N$ be arbitrary. Let $x, y \in X^N$. Then by Pareto and rationality, $(x_i) \succeq^0 \left((\succeq_i)_{i\in N}\right)(y_i)$ if and only if $\left(ce_{\succeq_i}(x_i)\right)_{i\in N} \succeq^0 \left((\succeq_i)_{i\in N}\right)\left(ce_{\succeq_i}(y_i)\right)_{i\in N}$. By independence $\left(ce_{\succeq_i}(x_i)\right)_{i\in N} \succeq^0 \left((\succeq_i)_{i\in N}\right)\left(ce_{\succeq_i}(y_i)\right)_{i\in N}$ if and only if $\left(ce_{\succeq_i}(x_i)\right)_{i\in N} \succeq^0 \left((\succeq_i')_{i\in N}\right)\left(ce_{\succeq_i}(y_i)\right)_{i\in N}$. Again by definition, $\left(ce_{\succeq_i}(x_i)\right)_{i\in N} \succeq^0 \left((\succeq_i')_{i\in N}\right)\left(ce_{\succeq_i}(y_i)\right)_{i\in N}$ if and only if $W\left(\left(ce_{\succeq_i}(x_i)\right)_{i\in N}\right) \geq W\left(\left(ce_{\succeq_i}(y_i)\right)_{i\in N}\right)$. $\square$

Our aim from this point on is to characterize those functions W which reduce uncertainty aversion.

Example 4: The simplest example of a W which reduces uncertainty aversion is given by

$$W(t_1, ..., t_n) = \max_{i\in N}\{t_i\}.$$

This function illustrates some of the properties of the reduction of uncertainty aversion. While it is true that the household rule generated by this function reduces uncertainty aversion on the domain $\mathcal{CP}$, the function W generates a very unfair rule. Moreover, the induced household preferences are typically not convex. If we denote by $U_i(c)$ the upper contour set of $\succeq_i$ at c, then it is easily verified that $U^*(c) = \bigcup_{i\in N} U_i(c)$ (this also verifies that this household preference is the most uncertainty averse preference which is less uncertainty averse than each individual in the household). In general, household preferences induced by maximal household welfare need not be convex. Thus, even if every individual in the household is uncertainty averse, the household need not be.

The following proposition illustrates that for any preference profile in $\mathcal{CP}$, any prior which is common to all agents is also a household prior. It holds because under the assumption $\mathcal{CP}$, constant allocations are always efficient, and constant allocations are always supported by the common prior. The intuition for the result is similar to results appearing in Billot, Chateauneuf, Gilboa, and Tallon (2000), Dana (2002) and Rigotti, Shannon, and Strzalecki (2008).

The c-simplex $\Delta_c = \left\{u \in \mathbb{R}^N_+ : \sum_N u_i = c\right\}$ is the set of nonnegative vectors summing to c.

Proposition 5: *Suppose that the conditions in Proposition 3 are satisfied. Let $(\succeq_i)_{i\in N} \in \mathcal{CP} \cap \mathcal{D}$ and suppose that the function W associated with $\succeq^0$ is strictly increasing. Then* $\bigcap_{i\in N} P(\succeq_i) \subseteq P\left(\succeq^*\left((\succeq_i)_{i\in N}\right)\right)$.

Proof: Let c be a constant act.

First, we show that constant allocations of c maximize $\succeq^0 \left((\succeq_i)_{i\in N}\right)$ across all allocations of c. Observe that for a constant act d_i, $ce_{\succeq_i}(d_i) = d_i$. Consequently, the constant allocation $(d_i)_{i\in N}$ maps to $(ce_{\succeq_i}(d_i))_{i\in N} = (d_i)_{i\in N}$, so that any vector on Δ_c is a vector of certainty equivalents for some constant allocation of c.

Now, let (y_i) be an allocation such that $\sum_{i\in N} y_i = c$. Note that, treating a certainty equivalent as a constant act, $p \cdot ce_{\succeq_i}(y_i) \leq p \cdot y_i$, as $ce_{\succeq_i}(y_i) \sim_i y_i$ and $p \in P(\succeq_i, ce_{\succeq_i}(y_i))$. Conclude that

$$\begin{aligned} \sum_{i\in N} ce_{\succeq_i}(y_i) &= \sum_{i\in N} p \cdot ce_{\succeq_i}(y_i) \\ &\leq \sum_{i\in N} p \cdot y_i \\ &= p \cdot c \\ &= c. \end{aligned}$$

Consequently, $(ce_{\succeq_i}(y_i))_{i\in N}$ lies below the Δ_c simplex, so there always exists a constant allocation $(d_i)_{i\in N}$ maximizing $\succeq^0 \left((\succeq_i)_{i\in N}\right)$ across all allocations of c.

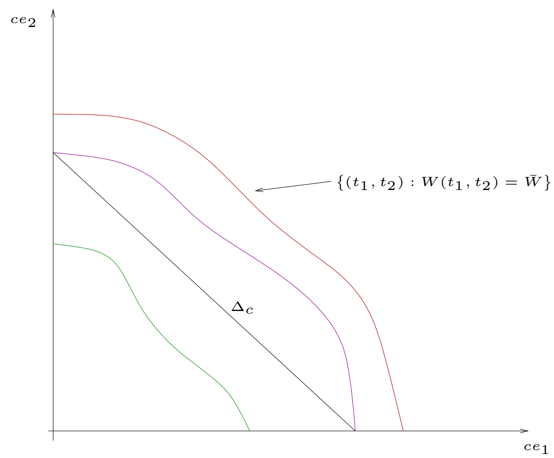


Figure 2: Level curves of W in the hypotheses of Theorem 6.

Secondly, let $p \in \bigcap_{i \in N} P(\succeq_i)$; we shall prove that $p \in P\left(\succeq^*\left((\succeq_i)_{i\in N}\right)\right)$. Let $x \succeq^* \left((\succeq_i)_{i\in N}\right) c$. Let $(x_i)_{i\in N}$ solve $\max W\left((ce_{\succeq_i}(x_i))_{i\in N}\right)$ subject to $\sum x_i = x$. Then by definition of $\succeq^*\left((\succeq_i)_{i\in N}\right)$, for all constant allocations $(d_i)_{i\in N}$ of c, $W\left((ce_{\succeq_i}(x_i))_{i\in N}\right) \geq W\left((d_i)_{i\in N}\right)$. Conclude that $\sum_{i\in N} ce_{\succeq_i}(x_i) \geq c$, as $\left((ce_{\succeq_i}(x_i))_{i\in N}\right)$ must lie outside of the c simplex. Now, $p \cdot c \leq p \cdot \left(\sum_{i\in N} ce_{\succeq_i}(x_i)\right) = \sum_{i\in N}(p \cdot ce_{\succeq_i}(x_i)) \leq \sum p \cdot x_i = p \cdot x.\square$

The following result is our main characterization theorem. It tells us that under our axioms, a social welfare function reduces uncertainty aversion if and only if it is associated with a function W which is maximized on any simplex at the vertices. Examples of such functions W are the max function as discussed above, and the sum (utilitarian) function $W(u) = \sum_{i\in N} u_i$. Other examples include the functions $W(u) = \left(\sum_{i\in N} u_i^p\right)^{1/p}$ for $p \geq 1$.

Theorem 6: *Suppose that $\mathcal{EU} \subseteq \mathcal{D} \subseteq \mathcal{CP}$. A social welfare function satisfies rationality, Pareto, independence, and reduction of uncertainty aversion on $\mathcal{D}$ if and only if there exists a strictly monotonic and continuous $W : \mathbb{R}^N_+ \to \mathbb{R}$, for which for each t-simplex and each $i \in N$, $(t_i, 0_{-i}) \in \arg\max_{\Delta_t} W$ such that for all $(\succeq_i)_{i\in N} \in \mathcal{R}^N$ and all $(x_i)_N, (y_i)_N \in X^N$,*

$$(x_i) \succeq^0 \left((\succeq_i)_{i\in N}\right)(y_i) \iff W\left((ce_{\succeq_i}(x_i))\right) \geq W\left((ce_{\succeq_i}(y_i))\right).$$

Figure 2 shows the level curves of a W in the hypotheses of Theorem 6. It should be intuitively clear that requiring W to be quasiconcave will pin down the sum of certainty equivalents. The following axiom contains the quasiconcavity restriction. It has the interpretation that in an uncertainty-free "divide the dollar" environment, social preferences should be "fair."

Quasiconcavity: *Let $(c_i)_{i\in N}$ and $(d_i)_{i\in N}$ be constant allocations. Let $(\succeq_i)_{i\in N} \in \mathcal{R}^N$. Suppose $(c_i)_{i\in N} \succeq^0 \left((\succeq_i)_{i\in N}\right)(d_i)_{i\in N}$. Then for all $\alpha \in [0,1]$, $(\alpha c_i + (1-\alpha)\, d_i)_{i\in N} \succeq^0 \left((\succeq_i)_{i\in N}\right)(d_i)_{i\in N}$.*

Corollary 7: *A rule $\succeq^0$ satisfies rationality, Pareto, independence, reduction of uncertainty, and quasiconcavity if and only if for all $(\succeq_i)_{i\in N} \in \mathcal{R}$ and all $x, y \in X^N$,*

$$(x_i)_{i\in N} \succeq^0 \left((\succeq_i)_{i\in N}\right)(y_i)_{i\in N} \Longleftrightarrow \sum_{i\in N} ce_{\succeq_i}(x_i) \geq \sum_{i\in N} ce_{\succeq_i}(y_i).$$

This corollary tells us that essentially the only "fair" SWF to reduce uncertainty aversion is the one which ranks allocations according to the sum of its certainty equivalents.

We end the section with a proof of Theorem 6.

Proof (Proof of Theorem 6): First, suppose there exists a W as in the statement of the theorem. Let $\left((\succeq_i)_{i\in N}\right) \in \mathcal{CP}$, and let $p \in \bigcap_{i\in N} P(\succeq_i)$. Let c be a constant act. We wish to show that for all $i \in N$, $\{x : x \succeq_i c\} \subseteq \left\{x : x \succeq^* \left((\succeq_i)_{i\in N}\right) c\right\}$.

As a first step, we show that for all $i \in N$, the allocation $(c, 0_{-i}) \in \arg\max_{\sum x_i = c} W\left((ce_{\succeq_i}(x_i))\right)$.

Let $(y_1, \ldots, y_n)$ be any allocation such that $\sum y_i = c$. We prove that there exists a constant allocation $(d_1, ..., d_n)$ of c such that $d_i \geq ce_{\succeq_i}(y_i)$ for all $i \in N$. This follows as in the proof of Proposition 5: any vector of certainty equivalents on Δ_c is achievable by a constant allocation: for a constant act d_i, $ce_{\succeq_i}(d_i) = d_i$. So any $(d_1, \ldots, d_n) \in \Delta_c$ with $\sum d_i = c$ is achieved by the constant allocation $(d_1, \ldots, d_n)$.

Second, by definition of p, $ce_{\succeq_i}(y_i) = p \cdot ce_{\succeq_i}(y_i) \leq p \cdot y_i$. Consequently,

$$\sum_{i\in N} ce_{\succeq_i}(y_i) \leq \sum_{i\in N} p \cdot y_i = p \cdot c = c.$$

Hence there is a vector $(d_1, \ldots, d_n) \in \Delta_c$ with $(ce_{\succeq_1}(y_1), \ldots, ce_{\succeq_n}(y_n)) \leq (d_1, \ldots, d_n)$. By the observation above, and monotonicity of W, the constant allocation $(d_1, \ldots, d_n)$ satisfies $W(y_1, \ldots, y_n) \leq W(d_1, \ldots, d_n)$. By the hypothesis on W, $W(c, 0_{-i}) \geq W(d_1, \ldots d_n)$; we therefore establish that $(c, 0_{-i}) \in \arg\max_{\Delta_c} W$.

Now, let $x \in X$ and suppose that $x \succeq_i c$. Then for all $(y_1, \ldots, y_n) \in X^N$ for which $\sum_{i \in N} y_i = c$,

$$W\left(ce_i\left(x\right), 0_{-i}\right) \geq W\left(c, 0_{-i}\right) \geq W\left(y_1, \ldots, y_n\right).$$

Therefore, for every allocation $(y_1, ..., y_n)$ of c, $(x, 0_{-i}) \succeq^0 \left((\succeq_i)_{i \in N}\right)(y_i)_{i \in N}$. By definition of $\succeq^* \left((\succeq_i)_{i \in N}\right)$, $x \succeq^* \left((\succeq_i)_{i \in N}\right) c$.

Conversely, suppose that $\succeq^0$ satisfies the axioms. W exists from Proposition 3; we will show that the vertices of every simplex maximize W on the simplex.

Fix a strictly positive $q \in \Delta(\Omega)$. We shall consider a profile $(\succeq_i)_{i \in N} \in \mathcal{EU}$ with common prior q. For $j \in N$, let

$$ce_{\succeq_j}\left(x\right) = q \cdot x$$

and for all $i \neq j$, choose some strictly increasing, concave, and differentiable $u_i : \mathbb{R}_+ \to \mathbb{R}$ for which $\lim_{x \to 0^+} u_i'(x) = +\infty$ and $\lim_{x \to +\infty} u_i(x) = 0$ (the Inada conditions) for which for all $x, y \in X$, and define $\succeq_i$ by

$$x \succeq_i y \Longleftrightarrow \sum_{\omega} q\left(\omega\right) u_i\left(x\left(\omega\right)\right) \geq \sum_{\omega} q\left(\omega\right) u_i\left(y\left(\omega\right)\right).$$

Note that $(\succeq_i)_{i \in N} \in \mathcal{EU}$ and that q is a common prior.

By Proposition 5, q is a prior for $\succeq^* \left((\succeq_i)_{i \in N}\right)$, and by reduction of uncertainty aversion, $\succeq^* \left((\succeq_i)_{i \in N}\right)$ must coincide with agent j's preference: to see this note that if $x \succeq^* ((\succeq_i)_{i \in N})c$ for a constant act c then $q \cdot x \geq c$ as the prior q supports c; thus $x \succeq_j c$. Then, by reduction of uncertainty aversion, we obtain that $\{x : x \succeq_j c\} = \{x : x \succeq^* ((\succeq_i)_{i \in N})c\}$. This implies that $ce_{\succeq_j} = ce_{\succeq^*((\succeq_i)_{i \in N})}$; thus $\succeq_j = \succeq^* \left((\succeq_i)_{i \in N}\right)$.

For each $i \in N$, define the **indirect utility function** $v_{\succeq_i} : \Delta(\Omega) \times \mathbb{R}_+$ by

$$v_{\succeq_i}\left(p, m\right) = \max_{p \cdot x \leq m} ce_{\succeq_i}\left(x\right).$$

Define $U_{\succeq^*\left((\succeq_i)_{i \in N}\right)} : X \to \mathbb{R}$ by

$$U_{\succeq^*\left((\succeq_i)_{i \in N}\right)}\left(x\right) = \sup_{\sum x_i = x} W\left(\left(ce_{\succeq_i}\left(x_i\right)\right)_{i \in N}\right).$$

Similarly, define

$$V_{\succeq^*\left((\succeq_i)_{i\in N}\right)}(p,m) = \max_{p\cdot x\leq m} U_{\succeq^*\left((\succeq_i)_{i\in N}\right)}(x).$$

By Chipman and Moore (1979), Theorem 3.9,

$$V_{\succeq^*\left((\succeq_i)_{i\in N}\right)}(p,m) = \max_{d\in\Delta(N)} W\left(\left(v_{\succeq_i}(p,d_i m)\right)\right).$$

By the Maximum Theorem, the correspondence $\delta : \Delta(\Omega) \times \mathbb{R}_+$ defined by

$$\delta(p,m) = \arg\max_{d\in\Delta(N)} W\left(\left(v_{\succeq_i}(p,d_i m)\right)\right)$$

is well-defined and upper semi-continuous. Define the demand correspondence $x_{\succeq}(p,m)$ as those allocations which are $\succeq$-maximal in the set $\{x : p\cdot x \leq m\}$.

By Chipman and Moore (1979), Corollary 3.5,

$$x_{\succeq^*\left((\succeq_i)_{i\in N}\right)}(p,m) = \bigcup_{d\in\delta(p,m)} \sum x_{\succeq_i}(p,d_i m).$$

Now, let $p \in \Delta(\Omega)$,$p \neq q$,$p \gg 0$. Since $\succeq^* \left((\succeq_i)_{i\in N}\right)$ coincides with $\succeq_j$, if $x \in x_{\succeq^*\left((\succeq_i)_{i\in N}\right)}(p,m)$, then if $\frac{p_\omega}{q_\omega} > \frac{p_{\omega'}}{q_{\omega'}}$, $x_\omega = 0$. Therefore, there exists ω for which $x_\omega = 0$. Moreover, for all $i \neq j$, if $m > 0$, $x_{\succeq_i}(p,m) \gg 0$. Consequently, we conclude that for all $d \in \delta(p,m)$, $d_i = 0$ for $i \neq j$. By upper semicontinuity of δ, conclude that $(1_j, 0_{-j}) \in \delta(q,m)$. Recall that

$$\delta(q,m) = \arg\max_{d\in\Delta(N)} W\left(\left(v_{\succeq_i}(q,d_i m)\right)_{i\in N}\right),$$

Note that for all i, $v_{\succeq_i}(q,d_i m) = d_i m$. Consequently $\left(v_{\succeq_i}(q,d_i m)\right)_{i\in N}$ lies on Δ_m. As $(1_j, 0_{-j}) \in \delta(q,m)$, we therefore conclude that $W(m_j, 0_{-j}) \geq W(u)$, for all $u \in \Delta_m$.

As j was arbitrary, the proof is complete. $\square$

4 Application: homothetic preferences

Theorem 6 gives a family of functions that reduce uncertainty aversion. We singled out the sum of certainty equivalents based on quasiconcavity. Here we provide another justification, one that holds for profiles of homothetic preferences.

The maximum function in Example 4 is the most uncertainty-averse preference which is less uncertainty averse than all individual preferences. As we remarked, this rule may in general induce non-convex household preferences: see Figure 1(b).

Here we look at the most uncertainty averse *convex* preference which is less uncertainty averse than all individual preferences. That is, we look for the household behavior that reduces uncertainty aversion while remaining within the domain of uncertainty averse preferences. We show that, for profiles of homothetic preferences, the sum of certainty equivalent gives the most uncertainty averse convex preference that is less uncertainty averse than members' preferences.

In Figure 1(b), the most uncertainty averse convex household is given by the convex hull of the two upper contour sets. If we denote the upper contour set of agent i's preference at c as $U_i(c)$, the upper contour set of the household preference at c is

$$\overline{\mathrm{co} \bigcup_{i \in N} U_i(c)},$$

the closed convex hull of the union of the individual upper contour sets. We shall prove this below.

Say a preference $\succeq \in \mathcal{R}$ is **homothetic** if for all $x, y \in \mathbb{R}_+^N$ and all $\alpha > 0$, $x \succeq y \Longrightarrow \alpha x \succeq \alpha y$. Denote the set of homothetic preferences by $\mathcal{H}$.

Example 8: Homothetic preferences are an important family of preferences in economic analysis. In particular, a large family of such preferences are given by the CRRA multiple priors agents, who have certainty equivalent utility representations $ce : X \to \mathbb{R}$ given by

$$ce(x) = \left(\min_{p \in P \subseteq \Delta(\Omega)} \int_\Omega [x(\omega)]^\rho \, dp(\omega) \right)^{\frac{1}{\rho}},$$

where $\rho \leq 1$. If, given a set of such agents with indices (P_i, ρ_i), the condition that the set of preferences has a common prior is equivalent to the condition that $\bigcap_{i \in N} P_i \neq \varnothing$.

Theorem 9: *Suppose that $(\succeq_i)_{i \in N} \in \mathcal{H}^N \cap \mathcal{CP}$. Consider the SWF represented by the sum of certainty equivalents. Then the household preference $\succeq^* \left((\succeq_i)_{i \in N}\right)$ is homothetic, and is the most uncertainty averse convex preference which is less uncertainty averse than $\succeq_i$ for all $i \in N$.*

The theorem demonstrates that at any constant act, the upper contour set of the household preference is the closed convex hull of the union of the individual upper contour sets. This means that the household preference generated by the sum of certainty equivalents is both tractable and geometrically simple. [3]

The proof demonstrates that every profile of common prior homothetic preferences has, for each agent, a representation as: $ce_{\succeq_i}(x) = \inf_{y \in C_i} x \cdot y$, where the common prior p minimizes $\sum_{\omega \in \Omega} y(\omega)$ in C_i. In particular, this set C_i can be explicitly calculated as

$$C_i = \{y : x \succeq_i 1 \Longrightarrow x \cdot y \geq_i 1\}.$$

Using this representation, it is easy to explicitly calculate household preference: it is given by $U_{\succeq^*\left((\succeq_i)_{i\in N}\right)}(x) = \inf_{y \in \bigcap_{i \in N} C_i} x \cdot y$.

Example 2 *Multiple priors: Suppose that for all $i \in N$, $ce_{\succeq_i}(x) = \min_{p \in P_i \subseteq \Delta(\Omega)} p \cdot x$. Then $x \succeq^* \left((\succeq_i)_{i\in N}\right) y \Longleftrightarrow \min_{p \in \bigcap P_i} p \cdot x \geq \min_{p \in \bigcap P_i} p \cdot y$. That is, a household of risk-neutral multiple priors decision makers behaves in the aggregate as a risk neutral decision maker whose set of priors is the intersection of the individual sets.*

Example 3 *CRRA expected utility maximizers: Suppose that for all $i \in N$, $ce_{\succeq_i}(x) = \left(\int_\Omega [x(\omega)]^{\rho_i} dp(\omega)\right)^{\frac{1}{\rho_i}}$ for $\rho_i \in [0,1]$. Then $x \succeq^* \left((\succeq_i)_{i\in N}\right) y \Longleftrightarrow \int_\Omega [x(\omega)]^{\max_{i\in N}\rho_i} dp(\omega) \geq \int_\Omega [y(\omega)]^{\max_{i\in N}\rho_i} dp(\omega)$.*

We end this section with a proof of Theorem 9

Proof (Proof of Theorem 9): The following two lemmas are well-known, but we reproduce them here for completeness.

> **Lemma 10:** *If $\succeq \in \mathcal{H}$, then the certainty equivalent is a utility representation for $\succeq$ which is homogeneous of degree one.*

Proof: $ce(x)$ is the the value of the constant act which is indifferent to x. Alternatively,

$$ce(x) = \inf\{c : c \succeq x\}.$$

To see that the certainty equivalent is homogeneous, let $x \in X$ and $\alpha > 0$. Then

$$\begin{aligned} ce(\alpha x) &= \inf\{\alpha c : \alpha c \succeq \alpha x\} \\ &= \alpha \inf\{c : \alpha c \succeq \alpha x\} \\ &= \alpha \inf\{c : c \succeq x\} \\ &= \alpha ce(x), \end{aligned}$$

[3] There is a similar result in the theory of international trade, on the maximization of profits under constant returns to scale and more than one industry. Lerner (1934) and Chipman (1966) present a "diagrammatic" argument.

where the second to last equality holds by homotheticity. $\square$

Lemma 11: *If u is monotone, homogeneous of degree one, and quasiconcave, then it is concave.*

PROOF: Let $x, y \in X$ and $\alpha \in [0,1]$. Suppose without loss of generality that $u(y) \geq u(x)$. If $u(x) = 0$, then by monotonicity,

$$\begin{aligned} u(\alpha x + (1-\alpha) y) &\geq u((1-\alpha) y) \\ &= (1-\alpha) u(y) = \alpha u(x) + (1-\alpha) u(y), \end{aligned}$$

verifying concavity. Otherwise, suppose $u(y) > 0$. Then

$$\begin{aligned} &\alpha u(x) + (1-\alpha) u(y) \\ = &\left[\left(\frac{\alpha}{\alpha + (1-\alpha) \frac{u(y)}{u(x)}} \right) u(x) + \frac{(1-\alpha) \frac{u(y)}{u(x)}}{\alpha + (1-\alpha) \frac{u(y)}{u(x)}} u\left(\frac{u(x)}{u(y)} y \right) \right] \\ &\times \left[\alpha + (1-\alpha) \frac{u(y)}{u(x)} \right]. \end{aligned}$$

Note now that $u(x) = \frac{u(x)}{u(y)} u(y) = u\left(\frac{u(x)}{u(y)} y \right)$, so that by quasiconcavity,

$$\leq u(x) \left[\alpha + (1-\alpha) \frac{u(y)}{u(x)} \right] = \alpha u(x) + (1-\alpha) u(y),$$

verifying concavity. $\square$

Let $(\succeq_i)_{i \in N} \in \mathcal{H}^N \cap CP$. By Lemmas 10 and 11, the certainty equivalent function $ce_{\succeq_i} : X \to \mathbb{R}$ is homogeneous and concave. Moreover, for all constant acts c, $ce_{\succeq_i}(c) = c$. Extend $ce_{\succeq_i}$ to all of $\mathbb{R}^N$ by defining

$$ce'_{\succeq_i}(x) = \begin{cases} ce_{\succeq_i}(x) & \text{if } x \geq 0 \\ -\infty & \text{otherwise} \end{cases}.$$

The function $ce'_{\succeq_i}$ is concave, monotonic, and upper semicontinuous. Its conjugate, $\left(ce'_{\succeq_i}\right)^* : \mathbb{R}^N \to \mathbb{R}$ is defined by

$$\left(ce'_{\succeq_i}\right)^*(x) = \inf_y x \cdot y - ce'_{\succeq_i}(x).$$

It is well-known that this function is itself concave and that there is a nonempty, closed, convex, upper comprehensive[4] set $C_i \subseteq \mathbb{R}^N_+$ for which

$$\left(ce'_{\succeq_i}\right)^* (x) = \left\{ \begin{array}{c} 0 \text{ if } x \in C_i \\ -\infty \text{ otherwise.} \end{array} \right\}.$$

Moreover,

$$ce'_{\succeq_i}(x) = \inf_{y \in C_i} x \cdot y.$$

(See, for example, Rockafellar (1970) Theorem 12.2 and Theorem 13.2). Now, let p be a common prior for the profile $(\succeq_i)_{i \in N}$. We claim that $p \in \bigcap_{i \in N} C_i$ and moreover that p lies on the boundary (has minimal sum) of each C_i. To see this, note that for each i and each constant act c, $ce'_{\succeq_i}(c) = c$, so $\inf_{y \in C_i} \sum_\omega y(\omega) = 1$. Now, suppose that $p \notin C_i$ for some C_i. In particular, by a standard separation argument, there exists $x \in \mathbb{R}^N_+ \setminus \{0\}$ for which $p \cdot x < \inf_{y \in C_i} y \cdot x$. Let c be a real number for which $p \cdot x < c < \inf_{y \in C_i} y \cdot x$. But then $x \succeq_i c$, while $p \cdot x < c$, contradicting the fact that p is a prior for $\succeq_i$. Now consider the function defined on X for which

$$U_{\succeq^*\left((\succeq_i)_{i \in N}\right)}(x) = \max_{\sum x_i = x} \sum ce_{\succeq_i}(x_i).$$

Clearly, this function can also be defined on all of $\mathbb{R}^N$, so that

$$U'_{\succeq^*\left((\succeq_i)_{i \in N}\right)}(x) = \max_{\sum x_i = x} \sum ce'_{\succeq_i}(x_i).$$

Moreover, it is easy to see, that since $U'_{\succeq^*\left((\succeq_i)_{i \in N}\right)}$ takes infinite values outside of X, for $x \in X$,

$$U_{\succeq^*\left((\succeq_i)_{i \in N}\right)}(x) = U'_{\succeq^*\left((\succeq_i)_{i \in N}\right)}(x).$$

Finally, as $U'_{\succeq^*\left((\succeq_i)_{i \in N}\right)}$ is the sup-convolution of the functions $\left(ce'_{\succeq_i}\right)_{i \in N}$, we conclude that the conjugate

$$\left(U'_{\succeq\left((\succeq_i)_{i \in N}\right)}\right)^* (x) = \inf_y x \cdot y - U'_{\succeq^*\left((\succeq_i)_{i \in N}\right)}(x)$$

[4]That is, if $x \in C$ and $y \geq x$, then $y \in C$.

is given by

$$\begin{aligned}\left(U'_{\succeq\left((\succeq_i)_{i\in N}\right)}\right)^*(x) &= \sum_N \left(ce'_{\succeq_i}(x)\right)^* \\ &= \left\{\begin{array}{c} 0 \text{ if } x \in \bigcap_{i\in N} C_i \\ -\infty \text{ otherwise.}\end{array}\right\}.\end{aligned}$$

See Rockafellar Theorem 16.4 and Corollary 16.4.1. Consequently,

$$U'_{\succeq^*\left((\succeq_i)_{i\in N}\right)}(x) = \inf_{y\in\bigcap_N C_i} \langle x, y\rangle .$$

Importantly for these arguments, $\bigcap_N C_i \neq \varnothing$, as each C_i is upper comprehensive and contains p. Hence, we conclude that household preference $\succeq^*\left((\succeq_i)_{i\in N}\right)$ is represented by

$$x \succeq^* \left((\succeq_i)_{i\in N}\right) z \Longleftrightarrow \inf_{y\in\bigcap_N C_i} \langle x, y\rangle \geq \inf_{y\in\bigcap_N C_i} \langle z, y\rangle ,$$

where for all i,

$$x \succeq_i z \Longleftrightarrow \inf_{y\in C_i} \langle x, y\rangle \geq \inf_{y\in C_i} \langle z, y\rangle .$$

Clearly, then, $\succeq^*\left((\succeq_i)_{i\in N}\right)$ is homothetic. To see that it is the most uncertainty averse convex preference which is less uncertainty averse than each individual preference, let c be a constant act. Note that $p \in \bigcap_{i\in N} C_i$ and also lies on the boundary of $\bigcap_{i\in N} C_i$ (it minimizes $\sum y(\omega)$ across $y \in \bigcap_{i\in N} C_i$). Consequently for any constant act c, $\inf_{y\in\bigcap_{i\in N} C_i} c\cdot y = c$. We will show that for any c,

$$\left\{x : x \succeq^* \left((\succeq_i)_{i\in N}\right) c\right\} = \overline{\mathrm{co}\bigcup_{i\in N}\{x : x \succeq_i c\}},$$

which will verify the result. So first, we show that for all $i \in N$, $\{x : x \succeq_i c\} \subseteq \left\{x : x \succeq^* \left((\succeq_i)_{i\in N}\right) c\right\}$. Note that $x \succeq_i c$ implies that for all $y \in C_i$, $x \cdot y \geq c$ which implies that for all $y \in \bigcap_{i\in N} C_i$, $x \cdot y \geq c$, which implies that $x \succeq^* \left((\succeq_i)_{i\in N}\right) c$. We therefore know that

$$\overline{\mathrm{co}\bigcup_{i\in N}\{x : x \succeq_i c\}} \subseteq \left\{x : x \succeq^* \left((\succeq_i)_{i\in N}\right) c\right\}$$

as $\succeq^* \left((\succeq_i)_{i\in N}\right)$ is upper semicontinuous and convex. Suppose now that there exists $w \in X$ such that $w \succeq^* \left((\succeq_i)_{i\in N}\right) c$, and for which $w \notin \overline{\text{co} \bigcup_{i\in N} \{x : x \succeq_i c\}}$. In particular there exists y for which, when normalized, $y \cdot w < c \leq y \cdot x$ for all i and all $x \succeq_i c$. We claim that for all $i \in N$, $y \in C_i$; otherwise, there would exist a separating vector (again nonnegative and normalized) z for which $y \cdot z < c < \inf_{y' \in C_i} y' \cdot z$. But then $z \succeq_i c$ and $y \cdot z < c$, contradicting $y \cdot x \geq c$ for all $x \succeq_i c$. Consequently, $y \in \bigcap_{i\in N} C_i$. Therefore, $\inf_{y \in \bigcap_{i\in N} C_i} y \cdot w < c$, so that $c \succ^* \left((\succeq_i)_{i\in N}\right) w$, a contradiction. $\square$

5 Discussion

5.1 Related literature

There is a vast literature on risk-sharing in economics. Seminal papers discussing optimal risk sharing include Borch (1962) and Wilson (1968) (see also Chateauneuf, Dana, and Tallon (2000)). Under the assumption that all agents are subjective expected utility maximizers, they determine that, under certain conditions (risk aversion or a continuum of states) all Pareto optimal allocations can be obtained by maximizing a weighted sum of subjective expected utilities.[5] A central result of Wilson (1968) is that the risk tolerance of household preference is the sum of risk tolerances of each individual *at the optimal household consumption*.[6]

We present our results in a framework with general "non-expected utility" preferences. Our theorem is general enough to apply to most decision theoretic models existing in the literature, including (but not limited to) Schmeidler (1989), Gilboa and Schmeidler (1989), Klibanoff, Marinacci, and Mukerji (2005), Maccheroni, Marinacci, and Rusti-

[5]In particular, under these assumptions, Pareto optimal allocations satisfy what Gollier (2001) terms the "mutuality" principle–consumption of each individual depends only on the aggregate amount in each state. As the sum of certainty equivalents satisfies the Pareto principle, any allocation it recommends is Pareto optimal and hence satisfies the mutuality principle.

[6]Gollier (2001) builds on this result, showing that if all individuals have identical preferences, then a weighted utilitarian planner who optimizes social welfare given a constraint on *average* consumption results in a less risk-averse household preference if and only if the individual risk tolerance is convex. It should be noted that this is a fixed-profile result: the weighted utilitarian rules applied to arbitrary subjective expected utility profiles do not typically reduce uncertainty aversion.

chini (2006), Machina and Schmeidler (1992), Ergin and Gul (2008), Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2008), and Seo (2008). In the paper, we have used the expression "uncertainty aversion," instead of risk aversion, to emphasize that we operate in a general framework of Knightian uncertainty.

The results hold when we restrict the domain to subjective expected utility preferences, preference profiles in $\mathcal{EU}$: this should be clear from the statements and proofs in Section 3. The results are also novel and interesting as results about $\mathcal{EU}$; but no additional insights or simplifications are gained from presenting them as results about $\mathcal{EU}$.

Our work differs from previous studies concerning Knightian uncertainty aversion in that it is normative. Most previous studies seek to explain behavioral phenomena in markets, for example see Dow and Werlang (1992), Epstein and Wang (1994), Epstein (2001), Rigotti and Shannon (2005) and Rigotti, Shannon, and Strzalecki (2008). In contrast, we try to understand the SWF that satisfy normatively appealing axioms. In that sense, the exercise is closer to Wilson (1968).

5.2 Conclusion

We study household preferences in the context of sharing risk and uncertainty. We are especially interested in household preferences that are less averse to uncertainty than the members' individual preferences.

Arguments for reduction in uncertainty aversion are familiar in economics, and appear as early as in Samuelson (1964), Vickrey (1964) and Arrow and Lind (1970). These arguments are normative: a collective *should* behave in a less uncertainty averse way. The arguments roughly say that less cautious collectives may reap the benefits of larger expected gains, and mitigate the risks by risk sharing.

We introduce two additional axioms: the Pareto criterion and that certain (sure) acts should be compared without regard for preferences over uncertainty. From the normative perspective, the Pareto criterion is obviously desirable, and the independence axiom should be appealing. Independence may not be appealing in a descriptive setting, in which any one agent can force a "breakdown" of negotiations; then attitudes toward uncertainty play a role even when the "optimal" choices feature no uncertainty (Rubinstein, Safra, and Thomson (1992) explain how attitudes toward uncertainty are important in a Nash bargaining context).

We characterize the household SWF which respect the Pareto criterion, compare certain acts without regard for preferences over uncertainty, and which reduce uncertainty aversion. The results single out the sum of certainty equivalents as the unique member of this class which is quasiconcave over certain allocations. Quasiconcavity, in turn, is a basic fairness requirement.[7]

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[7] Gorman (1959) argues that actual collectives may use a convex W: he believes that utility profiles that are not very unequal may be inherently stable. Any small advantage obtained by a group of agents will result in a political advantage, which will then reinforce the initially small advantage. The resulting collective will behave as if it used a convex social choice function.

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